Optimal Strategies in the Neighborhood of a Collison Course

S. Gutman*
NASA Ames Research Center, Moffett Field, Calif.
and
G. Leitmann†
University of California, Berkely, Calif.

We consider a simple differential game between pursuer P and evader E in the neighborhood of a nominal collision course. The payoff is the terminal lateral miss-distance. The control of each player is his acceleration normal to his velocity vector, and both players' controls are bounded. Saddlepoint strategies are deduced for three combinations of the acceleration bounds and are shown to be related to sgn $\dot{\sigma}$ where σ is the orientation of the line of sight (L.O.S.).

I. Problem Statement

We consider the problem of a pursuer P and an evader E maneuvering "near" a nominal collision course[1,2] (see Fig. 1). The situation at time t is shown in Fig. 2, where

$x_p \triangleq$ pursuer's position normal to the initial nominal line of sight
$x_E \triangleq$ evader's position normal to the initial nominal line of sight
$a_p \triangleq$ pursuer's acceleration normal to his nominal path
$a_E \triangleq$ evader's acceleration normal to his nominal path
$\sigma \triangleq$ orientation of the line of sight (L.O.S.)
$V_c \triangleq$ nominal closing speed
T $\triangleq$ nominal collision time
$V_p \triangleq$ pursuer's speed
$V_E \triangleq$ evader's speed
$\gamma_p \triangleq$ pursuer's velocity direction relative to his nominal path
$\gamma_E \triangleq$ evader's velocity direction relative to his nominal path

Referring to Fig. 2, if the deviations of P and E from their nominal positions P' and E', respectively, are small compared to the nominal range $V_c(T-t)$, then $\epsilon = x_E - x_p$. Letting $x_1(t) \triangleq \epsilon(t)$, $x_2(t) \triangleq \dot{\epsilon}(t)$, one has the equations of motion normal to the L.O.S.:

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = u(t) + v(t) \tag{1}$$

where

$$u = -a_p \cos\delta_0, \qquad v = a_E \cos\phi_0 \tag{2}$$

For $\dot{V}_p = \dot{V}_E = 0$ and small γ_p and $\gamma_E, a_p = V_p\dot{\gamma}_p, a_E = V_E\dot{\gamma}_E$. The normal accelerations are bounded; that is, $|V_p\dot{\gamma}_p| \le a_p^m, |V_E\dot{\gamma}_E| \le a_E^m$, with a_p^m and a_E^m given constants. Consequently,

$$|u| \le u^m \triangleq a_p^m |\cos\delta_0| \qquad |v| \le v^m \triangleq a_E^m |\cos\phi_0| \tag{3}$$

Since only displacements perpendicular to the line of sight influence its rotation, the relative position coordinate $\epsilon = x_E - x_p$ is of primary interest.[1,2] Thus, we are concerned with a differential game[3,4] with payoff

$$J = \tfrac{1}{2}\epsilon^2(T) - \tfrac{1}{2}\epsilon^2(0) = \int_0^T x_1(t) x_2(t)\,\mathrm{d}t \tag{4}$$

which P desires to minimize and E wishes to maximize.

We denote feedback controls (strategies) of P and E on (x,t)-space, R^3, by $p(\bullet)$ and $e(\bullet)$, respectively, where

$$u(t) = p[x(t),t], \qquad v(t) = e[x(t),t] \tag{5}$$

and we seek a saddlepoint strategy pair $\{p^*(\bullet), e^*(\bullet)\}$.

The measured variable (output) is the rate of the L.O.S. orientation $\dot{\sigma}$. For small deviations from the nominal positions, that is, small $\sigma(t)$,

$$\dot{\sigma}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\epsilon(t)}{V_c(T-t)} = \frac{x_1(t)}{V_c(T-t)^2} + \frac{x_2(t)}{V_c(T-t)} \tag{6}$$

Hence, eventually we shall need to express the saddlepoint strategies as functions of output $\dot{\sigma}$ only; we shall denote these strategies by $\hat{p}(\bullet)$ and $\hat{e}(\bullet)$, respectively.

II. Case a: $u^m > v^m$

Following Refs. 5 and 6, we define a decomposition $\{Y_i\}$ of (x,t)-space, R^3, shown in Fig. 3, by‡

$$Y_1 = \{(x,t): x_1 + x_2(T-t) - \tfrac{1}{2}\Delta\rho(T-t)^2 > 0\} \tag{7i}$$

$$Y_2 = \{(x,t): x_1 + x_2(T-t) + \tfrac{1}{2}\Delta\rho(T-t)^2 < 0\} \tag{7ii}$$

$$Y_3 = (Y_1 \cup Y_2)^c \tag{7iii}$$

where $\Delta\rho \triangleq u^m - v^m$.

The saddlepoint strategy pair is given by[5,6]

$$\left.\begin{aligned} p^*(x,t) &= -u^m \\ e^*(x,t) &= v^m \end{aligned}\right\} \quad \forall (x,t) \in \bar{Y}_1 \tag{8i}$$

$$\left.\begin{aligned} p^*(x,t) &= u^m \\ e^*(x,t) &= -v^m \end{aligned}\right\} \quad \forall (x,t) \in \bar{Y}_2 \tag{8ii}$$

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*Research Associate, National Research Council.
†Professor.

‡$(\;)^c$ denotes the complement of $(\;)$ on R^3.

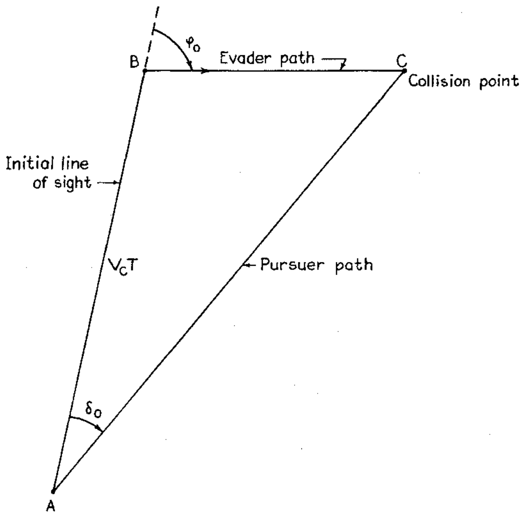

Fig. 1 Nominal collision course.

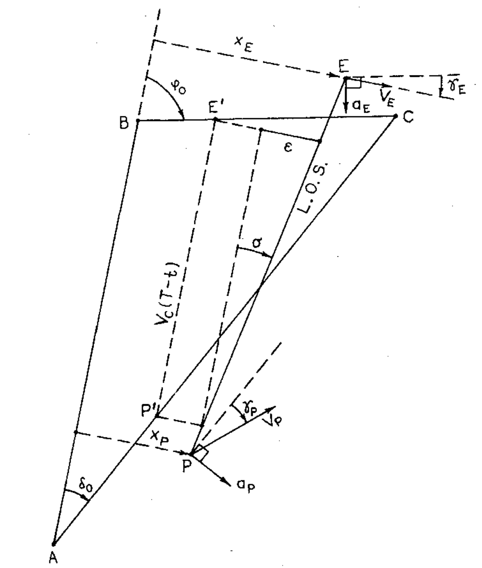

Fig. 2 Coordinate system for pursuit-evasion maneuver.

$\{p^*(x,t), e^*(x,t)\}$ = any admissible pair $\forall (x,t) \epsilon Y_3$. However, by (6),

$$x_1 + x_2(T-t) = V_c(T-t)^2 \dot{\sigma} \tag{9}$$

so that, for $t\epsilon[0,T)$, $\{Y_i\}$ induces a decomposition $\{S_i\}$ of R^1, defined by

$$S_1 = \{\dot{\sigma}:\dot{\sigma} > \Delta\rho/2V_c\},$$
$$S_2 = \{\dot{\sigma}\Sigma\dot{\sigma} < -\Delta\rho/2V_c\},$$
$$S_3 = \{\dot{\sigma}:|\dot{\sigma}| < \Delta\rho/2V_c\} \tag{7a}$$

The corresponding saddlepoint strategy pair $\{\hat{p}(\,\cdot\,), \hat{e}(\,\cdot\,)\}$ then is given by

$$\left.\begin{array}{l}\hat{p}(\dot{\sigma}) = -u^m \text{sgn}\dot{\sigma} \\ \hat{e}(\dot{\sigma}) = v^m \text{sgn}\dot{\sigma}\end{array}\right\} \quad \forall \dot{\sigma}\epsilon \bar{S}_1 \cup \bar{S}_2 \tag{8a}$$

$\{\hat{p}(\dot{\sigma}), \hat{e}(\dot{\sigma})\}$ = any admissible pair $\forall\, \dot{\sigma}\epsilon S_3$.

Figure 4 shows a block diagram from the pursuer's point of view of system (1) with output $\dot{\sigma}$ given by (9), pursuer feedback control $\hat{p}(\,\cdot\,)$ by (8a), and *linear* strategy on S_3, where the effective gain K_e is given by

$$u(t) = K_e V_c \dot{\sigma}(t), \qquad K_e \triangleq \frac{2}{1-(v^m/u^m)} \tag{10}$$

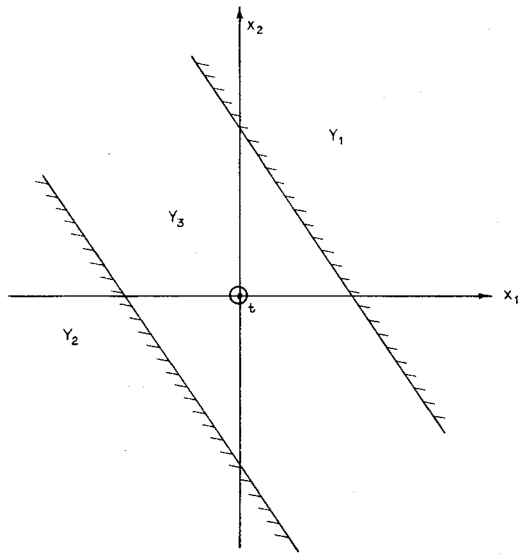

Fig. 3 Decomposition of (x,t)-space.

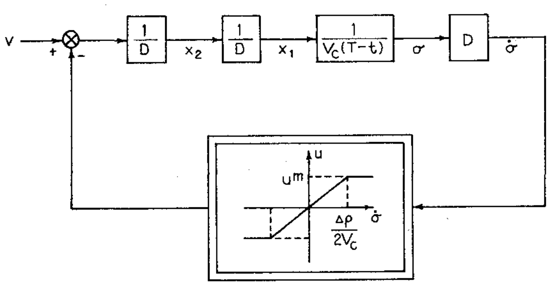

Fig. 4 Pursuer's strategy for Case a.

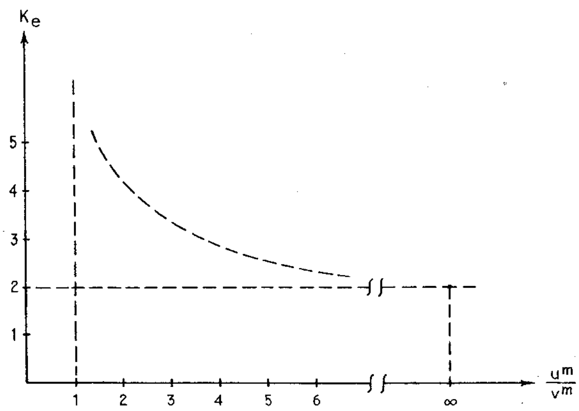

Fig. 5 Effective gain for Case a.

The proportional navigation gain K then is given by

$$\dot{\gamma}_P(t) = K\dot{\sigma}(t), \qquad K \triangleq K_e V_c / (V_p \cos\delta_0) \tag{10a}$$

The effective gain K_e as a function of relative maneuverability u^m/v^m is shown in Fig. 5.

III. Case b: $v^m > u^m$

Following Ref. 6, the decomposition $\{Y_i\}$ of (x,t)-space is defined by

$$Y_1 = \{(x,t): x_1 + x_2(T-t) > 0\} \tag{11i}$$

$$Y_2 = \{(x,t): x_1 + x_2(T-t) < 0\} \tag{11ii}$$

$$Y_3 = \bar{Y}_1 \cap \bar{Y}_2 \tag{11iii}$$

and that induced on $\dot{\sigma}$-space is then

$$S_1 = \{\dot{\sigma}:\dot{\sigma}>0\}, \quad S_2 = \{\dot{\sigma}:\dot{\sigma}<0\}, \quad Y_3 = \{\dot{\sigma}:\dot{\sigma}=0\} \qquad (12)$$

The corresponding saddlepoint strategy pair is given by

$$\left.\begin{aligned} \hat{p}(\dot{\sigma}) &= -u^m \operatorname{sgn}\dot{\sigma} \\ \hat{e}(\dot{\sigma}) &= v^m \operatorname{sgn}\dot{\sigma} \end{aligned}\right\} \quad \forall \dot{\sigma} \epsilon S_1 \cup S_2$$

$$\left.\begin{aligned} \hat{p}(\dot{\sigma}) &= -u^m \operatorname{sgn}\dot{\sigma} \\ \hat{e}(\dot{\sigma}) &= v^m \operatorname{sgn}\dot{\sigma} \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \hat{p}(\dot{\sigma}) &= u^m \operatorname{sgn}\dot{\sigma} \\ \hat{e}(\dot{\sigma}) &= -v^m \operatorname{sgn}\dot{\sigma} \end{aligned}\right. \quad \text{for } \dot{\sigma}=0 \qquad (13)$$

IV. Case c: $u^m = v^m$

Finally, we consider the case of equal maneuverability. Then, for $\{S_i\}$ defined by (12), it can be shown that $\{\hat{p}(\cdot),\hat{e}(\cdot)\}$ is given by[8]

$$\left.\begin{aligned} \hat{p}(\dot{\sigma}) &= -u^m \operatorname{sgn}\dot{\sigma} \\ \hat{e}(\dot{\sigma}) &= v^m \operatorname{sgn}\dot{\sigma} \end{aligned}\right\} \quad \forall \dot{\sigma} \epsilon S_1 \cup S_2 \qquad (14)$$

$\{\hat{p}(\dot{\sigma}),\hat{e}(\dot{\sigma})\}$ = any admissible pair for $\dot{\sigma}=0$.

V. Guaranteed Value of Terminal Miss

The guaranteed (saddlepoint) value of the payoff[5,6] as a function of initial conditions, is

Case a:

$$J^* = \tfrac{1}{2}T^4[V_c|\dot{\sigma}(0)| - \tfrac{1}{2}\Delta\rho]^2 - \tfrac{1}{2}\epsilon^2(0) \qquad \forall \dot{\sigma}(0)\epsilon S_1 \cup S_2$$

$$J^* = -\tfrac{1}{2}\epsilon^2(0) \quad \forall \dot{\sigma}(0)\epsilon S_3$$

Case b:

$$J^* = \tfrac{1}{2}T^4[V_c|\dot{\sigma}(0)| + \tfrac{1}{2}|\Delta\rho|]^2 - \tfrac{1}{2}\epsilon^2(0)$$

Case c:

$$J^* = \tfrac{1}{2}T^4[V_c\dot{\sigma}(0)]^2 - \tfrac{1}{2}\epsilon^2(0)$$

Hence, in view of (4), the corresponding guaranteed terminal miss is

Case a:

$$|\epsilon(T)| = T^2|V_c|\dot{\sigma}(0)| - \tfrac{1}{2}\Delta\rho| \quad \forall \dot{\sigma}(0)\epsilon S_1 \cup S_2$$

$$|\epsilon(T)| = 0 \qquad \forall \dot{\sigma}(0)\epsilon S_3$$

Case b:

$$|\epsilon(T)| = T^2[V_c|\dot{\sigma}(0)| + \tfrac{1}{2}|\Delta\rho|]$$

Case c:

$$|\epsilon(T)| = T^2V_c|\dot{\sigma}(0)|$$

VI. Remarks

1) For Case a, an arbitrary admissible strategy pair is a saddlepoint on S_3.

2) For Case a and linear strategy choice in S_3, the control characteristics shown in Fig. 4 agree with those used in practice.

3) Note the structural similarity between Eq. (10) here and (38) of Ref. 7. However, there are differences: a) the gain here is smaller than that of Ref. 7 by 2:3 and b) maneuverability here is measured in terms of the normal acceleration bounds a_P^m and a_E^m whereas in Ref. 7 it is measured by relative "energy capacity."

4) For equal maneuverability, Case c, the saddlepoint strategies are bang-bang.

5) for Case a and sufficiently small initial L.O.S. turning rate, $\dot{\sigma}(0)$, the pursuer can guarantee zero terminal miss, $\epsilon(T)=0$. However, $\epsilon(T)=0$ need not correspond to $\sigma(T)=0$; for instance, see the example on p. 180 of Ref. 1.

6) Finally, a word of caution is in order concerning the smallness assumption on which the analysis here as well as elsewhere[1,2,7] is based. Smallness of the deviations is relative to the nominal range $V_c(T-t)$ which goes to zero as the nominal collision time T is approached. Thus, the smallness assumption remains valid provided the deviations go to zero at least as rapidly as does the nominal range (i.e., at least linearly with time). Unless such behavior is verified in a particular case, all conclusions must be taken *cum grano salis*.

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